Serial Number 33

THE COLLEGES OF OXFORD UNIVERSITY

Entrance Examination in Mathematics

MATHEMATICS II

14 November 1995 Morning

Time allowed: 3 hours

Answers to each of Sections A, B, C and D must be attached to separate cover sheets and handed in separately. If no questions are attempted in any one section the cover sheet should still be handed in. Each cover sheet should be clearly labelled A, B, C or D.

All questions carry the same mark. There is no restriction on the number of questions any candidate may attempt but candidates are expected to attempt at least **three** *questions. Only the best* **five** *solutions will contribute to the total mark for the paper.*

The use of calculators is allowed, but, unless otherwise stated, exact answers should be given.

SECTION A

A1. (a) Show that the complex number w is real if and only if $|w-i| = |w+i|$, where $i^2 = -1$.

(b) Let C be the set of complex numbers $z \neq i$ such that $z_1 = \dfrac{iz}{i-z}$ is real. Giving reasons, describe C as a curve in the Argand diagram.

(c) Show that, for z in C, the points in the Argand diagram representing i, z and z_1 lie on a straight line.

A2. The points P and Q on the parabola $y^2 = 4ax$ have co-ordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$, where $p \neq q$, and A is the point with co-ordinates $(a, 0)$.

(a) Find the gradient of the line AP.

(b) Find the gradient of the line PQ.

(c) Find the gradient of the tangent to the parabola at P.

(d) Find the co-ordinates of the point of intersection T of the tangents to the parabola at P and Q.

(e) Find the co-ordinates of the point Q in terms of p when the line PQ passes through A.

A3. Three distinct points B, C and D in the plane have position vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ with respect to the origin. Show that B, C and D lie on a straight line if and only if

$$\lambda\mathbf{b} + \mu\mathbf{c} + \nu\mathbf{d} = \mathbf{0},$$

for some real numbers λ, μ and ν, all non-zero, and satisfying

$$\lambda + \mu + \nu = 0.$$

Find an additional condition on λ, μ and ν which will ensure that D also lies between B and C.

When this is the case show that D divides BC in the ratio $\beta : \gamma$ if and only if

$$\mathbf{d} = (\gamma\mathbf{b} + \beta\mathbf{c})(\beta + \gamma)^{-1}.$$

A4. (i) The line $x + y = 1$ in the Cartesian plane divides the disc with centre the origin and unit radius into two parts. By means of a diagram, find the area of each part.

(ii) Two discs S_1 and S_2 in the plane are of unit radius and have boundary circles which intersect at right angles. Giving reasons, find the area of their union $S_1 \cup S_2$.

(iii) Two discs D_1 and D_2 in the plane are both of unit radius and have centres C_1 and C_2. Their boundary circles intersect in the points A and B, where $\angle AC_1B = \pi/3$. Giving reasons, find the area of their union $D_1 \cup D_2$.

[The union $X \cup Y$ of two sets X and Y is the set consisting of all points which are in X or in Y or in both X and Y.]

SECTION B

B5. A ball of mass m is attached to the ground by a light elastic string of natural length l and modulus of elasticity λ. The ball is projected vertically upwards with speed $V > \sqrt{2gl}$ from the point at which the string is attached to the ground and, after time t, the ball is at a height y above the ground.

(a) Show that, for $0 \leq y \leq l$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -g$$

and find the corresponding equation when $y > l$.

(b) Show that, for $0 \leq y \leq l$,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + 2gy = V^2$$

and, for $y > l$,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + 2gy + \frac{\lambda}{ml}(y-l)^2 = V^2.$$

(c) Find the maximum height that the ball reaches and show that the tension in the string at the maximum height is

$$mg\left(-1 + \sqrt{1 + \frac{2\lambda}{mg}\left(\frac{V^2}{2gl} - 1\right)}\right).$$

B6. An archer shoots an arrow with velocity V at an angle α to the horizontal and the arrow just clears a wall. The horizontal distance of the wall from the archer is d and the height of the wall above that of his bow is h. Air resistance may be ignored.

(a) Show that

$$\frac{gd^2}{2V^2}\tan^2\alpha - d\tan\alpha + h + \frac{gd^2}{2V^2} = 0.$$

(b) Deduce that V must satisfy the inequality

$$V^2 - \frac{g^2 d^2}{V^2} \geq 2gh.$$

(c) Show that the minimum velocity required for the arrow to clear the wall is

$$\sqrt{g(h + (d^2 + h^2)^{1/2})},$$

and find the corresponding angle of projection.

B7. A particle P of mass m is slightly disturbed from rest at the highest point on the outside of a fixed smooth sphere of radius a and centre O. As long as the particle remains on the surface of the sphere, after time t, the angle between OP and the upward vertical is θ.

(a) Show that

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{g}{a}\sin\theta$$

and that the normal reaction exerted by the sphere on the particle is

$$m\left(g\cos\theta - a\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right).$$

(b) Show that the particle loses contact with the sphere when $\cos\theta = 2/3$.

(c) If, instead of being released from the highest point, the particle is released from rest at the point for which $\cos\theta = 2/3$, for what value of θ does the particle lose contact with the sphere?

B8. A ladder of length l and mass m rests with its lower end on rough horizontal ground and with its upper end against a smooth vertical wall. The centre of mass of the ladder is located at its mid-point and a painter of mass M stands on the ladder. Show that the ladder will not slip if the coefficient of friction μ between the ladder and the ground satisfies the inequality

$$\mu > \frac{(mh + 2MH)}{2(m+M)}\frac{\sqrt{l^2 - h^2}}{h^2},$$

where h and H are, respectively, the height of the upper end of the ladder and the height of the painter's feet above the ground.

Deduce that, no matter where the painter stands on the ladder, the ladder will not slip if

$$\mu > \cot\theta,$$

where θ is the angle between the ladder and the horizontal.

Turn Over

SECTION C

C9. Xanadu is a game in which a player's turn consists of throwing six fair dice. Alice is playing and in her first turn she gets exactly two sixes.

(a) Show that the probability of this is

$$\frac{1}{2}\left(\frac{5}{6}\right)^5.$$

(b) A player wins a small prize if among the six numbers thrown there is at least one pair the same. Find the probability of this.

(c) Bob wins a small prize in each of his first fifty turns. What is the expected number of times he should win in fifty turns? If you were playing against him would you be suspicious? Give reasons for your answer.

(d) A player wins a big prize if in the six dice thrown there are at least four sixes. On her second throw Alice does not look at the dice. She is told she has won a small prize. What is the probability that she has also won a big prize?

C10. In the game of *antidarts* a player shoots an arrow into a rectangular board measuring 6 metres by 8 metres. If the arrow is within 1 metre of the centre it scores 1 point, between 1 and 2 metres away it scores 2, between 2 and 3 metres it scores 3, between 3 and 4 metres and on the board it scores 4, and further than 4 metres but on the board it scores 5. William Tell always lands his arrows on the board but otherwise they are purely random.

(a) Show that the probability that his first arrow scores more than 3 points is $1-(3\pi/16)$.

(b) Find the probability that he scores a total of exactly 4 points in his first two arrows.

(c) Show that the probability that he scores exactly 15 points in three arrows is given by

$$\left(1-\frac{2}{3}\sin^{-1}\left(\frac{3}{4}\right)-\frac{1}{8}\sqrt{7}\right)^3.$$

C11. The national lottery of Cuckooland awards a prize to anyone who selects the same four integers as the robot Orwell when he selects four balls without replacement from a bag containing ten balls numbered 0,1,2,...,9.

(a) What is the probability that Orwell picks 0,1,2,3 in order?

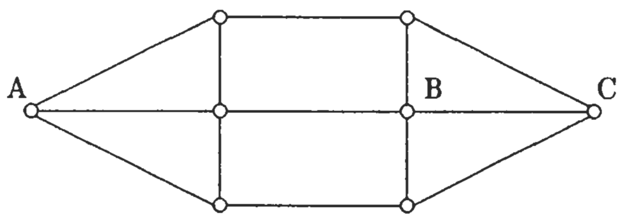
(b) Find the probability that the four numbers picked by Orwell are all greater than 5.

(c) What is the probability that 0 is among the four numbers picked by Orwell?

(d) What is the probability that at least one of the numbers picked by Orwell is greater than 7?

(e) For three successive weeks all the numbers picked by Orwell are odd. Calculate the probability of this happening.

C12. A mole has a network of burrows as shown. Each night he sleeps at one of the junctions. Each day he moves to a neighbouring junction but he chooses a passage randomly, all choices being equally likely from those available.



(a) He starts at A. Find the probability that two nights later he is at B.

(b) Having arrived at B, find the probability that two nights later he is again at B.

(c) A second mole is at C at the same time as the first mole is at A. What is the probability that two nights later the two moles share the same junction?

SECTION D

D13. The numbers l, m, n are positive integers such that $l \geq m \geq n$ and

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1.$$

Prove that $n \leq 3$ and $m \leq 4$. Hence find all possibilities for l, m, n.

What are the possibilities if

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1?$$

D14. (a) Let $a, b_1, \ldots, b_n$ be integers, and write $b = \max(b_1, b_2, \ldots, b_n)$ for the maximum of $b_1, b_2, \ldots, b_n$. Explain very briefly why

$$b_i = \min(b_i, b) \quad \text{for } i = 1, 2, \ldots, n,$$

and show that

$$\min(a, b) = \max\Big(\min(a, b_1), \min(a, b_2), \ldots, \min(a, b_n)\Big).$$

(b) Each of the colleges of the University of Macdorf has two mascots, which by ancient tradition are always an Owl and a Pussycat. The quango set up to investigate such matters has discovered that the mascots satisfy the following two conditions.

i. The age of the oldest of the college Owls is the same as the age of the oldest of the college Pussycats.

ii. If any two of the colleges were to swap Owls, then the younger mascot in each of the two colleges would have the same age as the younger mascot in the other college.

Given that none of the swaps actually takes place, prove that in each college the Owl and the Pussycat have the same age.

D15. For reasons of aesthetic efficiency Brussels decrees in the year 2001 that throughout the European Union only trains conforming to the following regulations shall be legal.

Rule 1. The engine shall be painted black (B), and each wagon shall be painted either gold (G) or red (R).

Rule 2. A train BR consisting of a black engine followed by a single red wagon shall be legal.

Rule 3. Given a legal train with n wagons, it shall be legal to attach to its rear a further n wagons which repeat the same pattern of colours; thus if BRRG is legal, so is BRRGRRG.

Rule 4. Given a legal train, it shall be legal (i) to attach a gold wagon to its rear provided that its last wagon is red, or (ii) to remove two consecutive wagons provided that they are both gold, or (iii) to replace three consecutive wagons by a single gold wagon provided that all three are red; thus if BRRRGGR is legal, so are (i) BRRRGGRG, (ii) BRRRR, and (iii) BGGGR.

(a) Prove that the trains BR^8G, BR^5, BR^7 are all legal, where R^k stands for k consecutive wagons, all red.

(b) Show that, in plain English, what the regulations mean is that a train whose only engine is at the front will be legal if and only if it satisfies Rule 1 and the number of red wagons is not a multiple of 3. [You may find it helpful to start by considering trains without gold wagons.]

D16. (a) Show that the equation $x^2 - 4x + 2 = 0$ has roots α, β such that $\alpha > 1 > \beta > 0$.

(b) For $n = 0, 1, 2, \ldots$, let $I(n) = \alpha^n + \beta^n$. For each n, show that

$$I(n+2) = 4I(n+1) - 2I(n),$$

and deduce that $I(n)$ is always an integer.

(c) For each n, show that $I(n)$ is in fact the least integer such that $I(n) > (2+\sqrt{2})^n$.